Percolation in random-Sierpiński carpets: A real space renormalization group approach

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The site percolation transition in random Sierpiński carpets is investigated by real space renormalization. The fixed point is not unique like in regular translationally invariant lattices, but depends on the number k of segmentation steps of the generation process of the fractal. It is shown that, for each scale invariance ratio n, the sequence of fixed points $p_{n,k}$ is increasing with k, and converges when $k\rightarrow\infty$ toward a limit p_n strictly less than 1. Moreover, in such scale invariant structures, the percolation threshold does not depend only on the scale invariance ratio n, but also on the scale. The sequence $p_{n,k}$ and p_n are calculated for $n=4$, 8, 16, 32, and 64, and for $k=1$ to $k=11$, and $k=\infty$. The corresponding thermal exponent sequence $\nu_{n,k}$ is calculated for $n=8$ and 16, and for $k=1$ to $k=5$, and $k=\infty$. Suggestions are made for an experimental test in physical self-similar structures. [S1063-651X(96)04510-2]

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I. INTRODUCTION

Phase transitions in fractals are of special interest since they are structures of noninteger dimension in which the critical behavior can be compared with the analytical continuations obtained from the renormalization group methods.

Transitions in finite ramification order fractals (i.e., in which any bounded part of the structure can be isolated by cutting a finite number of bonds) have been extensively studied. Gefen, Aharony, and Mandelbrot [1] proved in this case that transitions occur only at zero temperature. Then the Sierpiński gasket model (triangular self-invariant fractal) has been solved exactly for the Potts model, the Ising model, percolation, and electric conductance (Gefen *et al.* [2]). More recently, Yang [4] found an exact expression for the partition function of the Ising model on some Sierpiński carpets with finite ramification order.

However, few results are known on infinite ramification order fractal lattices, those in which transitions can occur at a nonzero temperature. Gefen, Aharony, and Mandelbrot [3] found some approximations for the critical exponents in Sierpiński carpets of various fractal dimension, for the Ising model, and the conductivity transition, using real space renormalization methods. It appears that the critical exponents depend not only on the fractal dimension, but also on other geometrical parameters like lacunarity and connectivity. Thus the self-invariant lattices do not follow the universal behavior observed in translationally invariant lattices.

Interesting questions arise from these works: (i) what is the role of the fractal dimension in the phase transition? (ii) What are the relevant parameters and what are their physical meaning? (iii) How does the renormalization group work when translation invariance is replaced by scale invariance?

This paper is a contribution to these questions. We investigate one of the most simple second order phase transition: the percolation transition, in a general class of fractals: the random Sierpiński carpets (RSC) (defined later) using a real space renormalization group method.

Percolation is a geometrical second order phase transition occurring in random lacunary media. Considering a set of elements randomly distributed on the sites of a lattice, the percolation transition is the property of these elements to become connected in a cluster of infinite size when their concentration is large enough. The concentration at which such connections occur is called the percolation threshold. The real space renormalization group has been extensively used to study the critical properties of the percolation transition in regular (i.e., translationally invariant) lattices, leading to the successful calculation of several physical characteristics of lacunary materials, such as their dielectric functions [5]. Few works, however, have been devoted to the percolation transition in random fractals which are a special class of lacunary lattices frequently used as models for disordered materials.

Random Sierpiński carpets (RSC) are a general class of regular-random fractals [7] generated by a segmentation process like the well known Sierpiński carpet, but generalized to any scale invariance ratio n and to a random choice of q conserved subsquares among the n^2 generated at each segmentation step. They are diluted fractal lattices quite different from the diluted regular square lattice. Indeed many configurations which occur into the diluted square lattice do not exist in a scale invariant fractal structure with the same concentration. We thus expect a quite different behavior of the percolation parameters, as already suggested by previous works [6].

Here, the critical properties of the percolation transition into RSC are investigated with a real space renormalization group method. More generally, this paper brings some new insights about the relation between renormalization group and fractality, both involving scale invariance, but with a different point of view. From the combination of these two related but somewhat different aspects of the scale invariance rise some interesting percolation properties of scale invariant fractal structures which should find experimental applications as suggested in the conclusion.

II. THE FRACTAL STRUCTURES

RSC are built as follows. An initial square is divided into n^2 subsquares, only q of them are conserved at random. This

FIG. 1. Two examples of random Sierpiński carpets: (a) RSC(4,11,3) ($d_f=1,73$) and (b) RSC(8,45,2) ($d_f=1,83$).

random segmentation is repeated on each conserved subsquares, and so on, k times (k is the segmentation step). Such a set is called RSC(n,q,k). The rigorous mathematical fractal is obtained after an infinite number of segmentation steps k. However, it can be observed in most ''natural'' fractals that the physical consequences of scale invariance appear with a small number of segmentation steps. Figure 1 shows two examples: RSC(4,11,3) (fractal dimension $d_f=1,73$) and RSC(8,45,2) ($d_f=1,83$). The concentration of occupied sites at step k is $(q/n^2)^k$ and tends to 0 for increasing values of k. This is a typical consequence of scale invariance, and a proof that for such fractals, the concentration cannot be used as a control parameter for the percolation transition. We will use instead $(q/n^2)^k$, the concentration at the first segmentation step, $p=q/n^2$ here called the ''initial concentration'' of RSC(n,q,k), which seems to be a better control parameter.

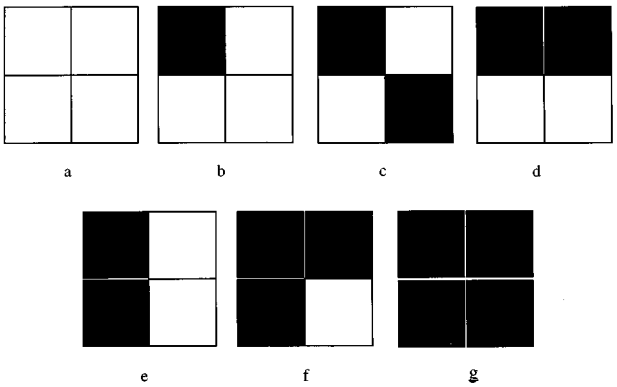

FIG. 2. The seven configurations of a four sites cell. Black squares are occupied sites and white squares are empty sites.

III. THE RENORMALIZATION PROCESS

The renormalization process is the one used previously to calculate the critical parameters of site percolation in regular lattices [8]. It corresponds to the ''one cell approximation'' [10, 11]. The cells are blocks of four sites. Seven configurations are distinguished for each cell, as shown in Fig. 2, according to the fact that sites are occupied (black squares) or empty (white squares). The renormalization rules are the following: four occupied squares (configuration g), three occupied squares (configuration f) and two occupied squares only if aligned on an arbitrary choosen direction (here vertical) give an occupied site. All other configurations (a, b, c, d) give an empty site in the renormalized lattice.

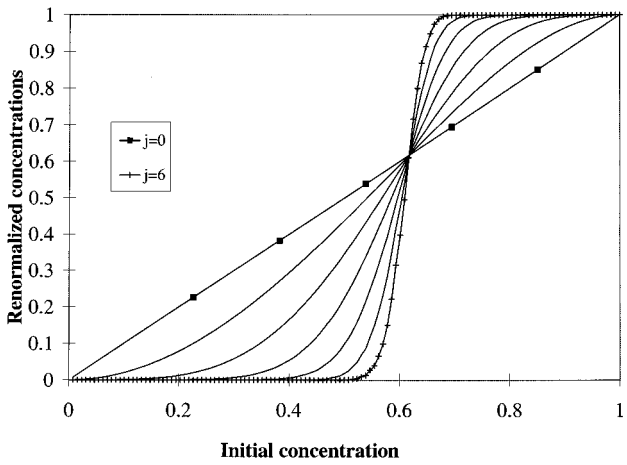

FIG. 3. Plot of the concentrations sequences of the renormalized regular lacunary square lattice of 4096 sites, vs the initial concentration $p=q/4096$. j is the renormalization step.

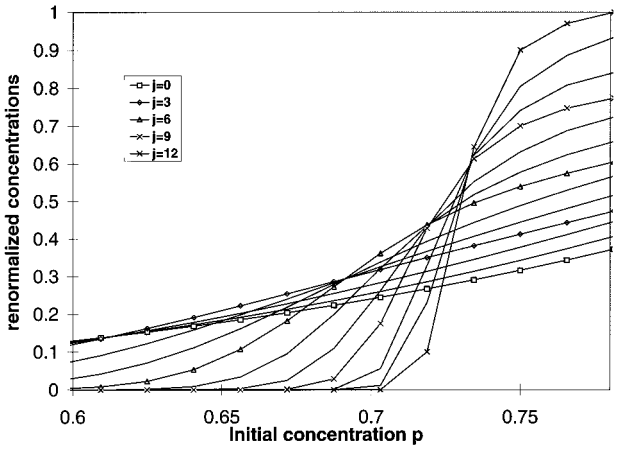

FIG. 4. Plot of the concentrations sequences of the renormalized lattices vs the initial concentration $p=q/64$, for RSC(8,q,4), showing the four fixed points. j is the renormalization step.

In lattices of finite size (finite number of segmentation steps for RSC), no infinite percolation cluster can exist. A physical way to analyze the percolation transition is to look for a continuous path between two arbitrarily choosen opposite sides of the lattice [8]. This explains the symmetry breaking involved by the distinction between horizontal lines and vertical lines in the renormalization process (configurations d and e in Fig. 2).

In the infinite regular lacunary square lattice, the renormalization function ϕ is easily calculated. Let us call p the concentration. The fractions of the three percolating cells (e, f, and g, in Fig. 2), obtained after one renormalization step are, respectively: $p^2(1-p)^2$, $p^3(1-p)$, and p^4. Taking into account the number of occurrences of each cell: two for e, four for f, and one for g, the renormalized concentration after one step, is $\phi(p)=p^2(2-p^2)$. The fixed point of this renormalization function is an estimation of the percolation threshold p_c. It is the nontrivial and positive solution of the equation: $p^2(2-p^2)-p=0$, that is $p_c=(\sqrt{5}-1)/2 = 0.618\,034$. This result is a reasonably good approximation of the percolation threshold of the square lattice obtained by Monte Carlo simulations: $p_c \simeq 0.59$ [9].

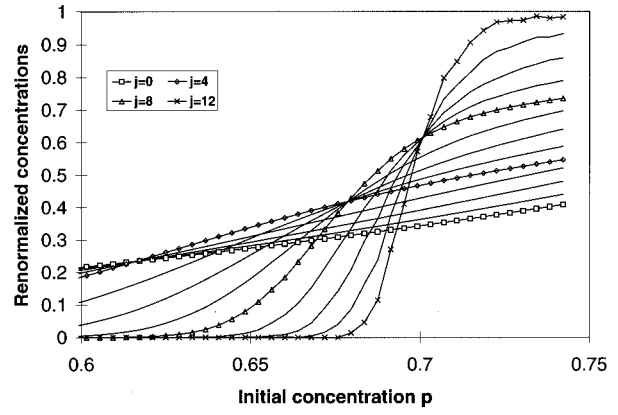

FIG. 5. Plot of the concentrations sequences of the renormalized lattices vs the initial concentration $p=q/256$, for RSC(16,q,3), showing the three fixed points. j is the renormalization step.

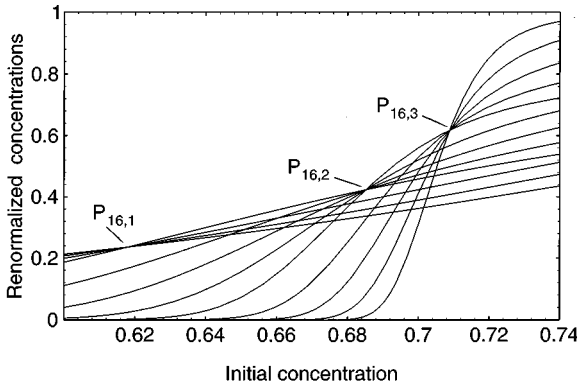

FIG. 6. Plot of the renormalized concentrations sequences $\psi_l(p)$ $(1 \leqslant l \leqslant k \times i = 12)$ vs the initial concentration $p=q/64$, for RSC(8,q,4) (to be compared to numerical results of Fig. 4).

IV. NUMERICAL SIMULATIONS

In the regular lacunary square lattice, successive applications of the renormalization described above on a 64×64 lattice give a set of six curves as shown in Fig. 3 (j is the renormalization step). All curves intersect at the fixed point of the renormalization group transformation, corresponding to p_c (neglecting the finite size effects).

For RSC(n,q,k), the same renormalization process is applied on structures with a scale invariance ratio n which is a power of 2: $n=2^i$ in order to get, at each renormalization step, an even number of sites near each edge of the lattice. The total number of renormalization steps is $k\times i$. The case $k=1$ corresponds exactly to the percolation problem on the regular square lattice of size n^2.

Results are shown in Fig. 4 for RSC(8,q,4), and Fig. 5 for RSC(16,q,3). The sequences of concentrations obtained by iterations of the renormalization process are plotted versus the initial concentration (j is the renormalization step). The renormalized concentrations are averaged over 200 randomly choosen structures for RSC(8,q,4), and 500 for RSC (16,q,3) to avoid fluctuations effects. It appears that, compared to the lacunary square lattice, there is not a unique fixed point, but a sequence of fixed points, depending on the segmentation step of the fractals. The main result is that the percolation threshold depends on the scale at which the fractal is observed.

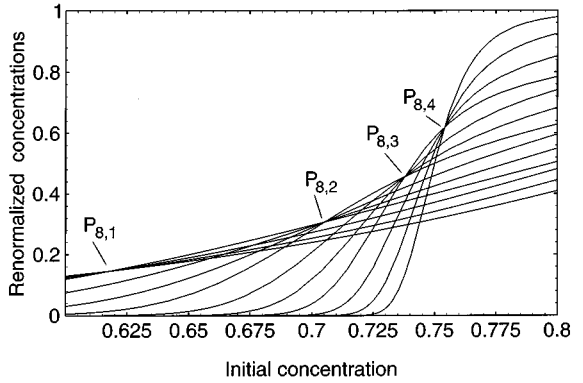

FIG. 7. Plot of the renormalized concentrations sequences $\psi_l(p)$ $(1 \leqslant l \leqslant k \times i = 12)$ vs the initial concentration $p=q/256$, for RSC(16,q,3) (to be compared to numerical results of Fig. 5).

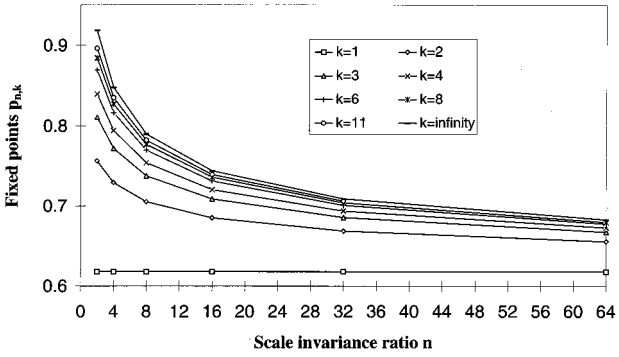

FIG. 8. Plot of the fixed points $p_{n,k}$ of RSC(n,q,k) vs n, for $k\leqslant 1\leqslant 11$ and $k=\infty$.

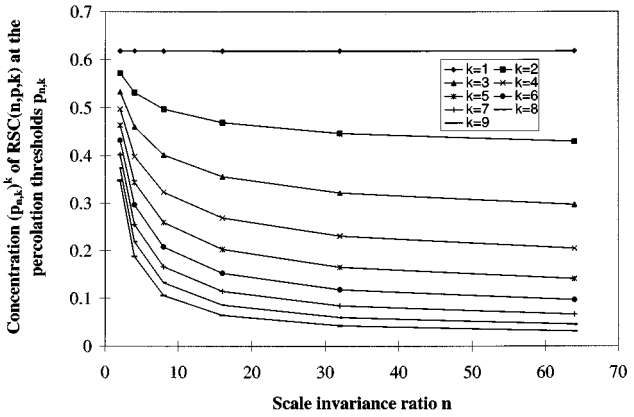

FIG. 9. Plot of the concentrations $p_{n,k}^k$ of RSC(n,q,k) at the percolation thresholds $p_{n,k}$ of RSC(n,q,k) vs n, for $k\leqslant 1\leqslant 9$.

Three other important observations arise from these numerical simulations. (i) The number of fixed points is exactly equal to the number k of segmentation steps; (ii) each fixed point is the intersection of the same number $i+1$ of renormalization curves; (iii) the first k fixed points of RSC$(n,q,k+1)$ are the same as the k fixed points of RSC(n,q,k), i.e., increasing the number of segmentation steps only adds new fixed points without changing the values of the preceding ones.

V. ANALYTICAL CALCULATIONS

The calculation of the renormalized concentrations at each step of renormalization can be performed in the following way. The important fact is that we have choosen values of n such that the size of the lattice is a multiple of two at each renormalization step $(n=2^i)$. As RSC$(n,q,1)$ (with only one step of segmentation) is the same structure as a random lacunary square lattice with size n^2 with a concentration $p=q/n^2$, RSC(n,q,k) can be considered as a set of $q^{2(k-1)}$ lacunary square lattices each with concentration q/n^2. As $n=2^i$, the renormalization occurs independently into the $q^{2(k-1)}$ random lacunary square lattices up to the ith step of renormalization, by i successive applications of the renormalization function ϕ. Let us call ϕ_j the jth iterate of the function ϕ, and $\psi_j(p)$ the function giving the concentration at the jth step of iteration of the renormalization process. Starting with the concentration $\psi_0(p)=p^k=(q/n^2)^k$, the i first renormalization steps give the following sequence of renormalized concentrations:

$$\psi_0(p)=p^k,$$

$$\psi_1(p)=p^{(k-1)}\phi(p),$$

$$\psi_2(p)=p^{(k-1)}\phi_2(p), \tag{5.1}$$

$$\cdots\cdots\quad\cdots\cdots\quad\cdots\cdots$$

$$\psi_i(p)=p^{(k-1)}\phi_i(p).$$

At this step, the renormalized structure obtained is a set of $q^{2(k-1)}$ squares, each with the renormalized concentration $\phi_i(p)$. It can be considered as a set of $q^{2(k-2)}$ random lacunary square lattices of size n^2, each with concentration $p\times\phi_i(p)$. The i next renormalization steps are then obtained by action of the renormalization function ϕ, applied i times on this concentration $p\times\phi_i(p)$ giving

$$\psi_{(i+1)}(p)=p^{(k-2)}\phi[p\,\phi_i(p)],$$

$$\psi_{(i+2)}(p)=p^{(k-2)}\phi_2[p\,\phi_i(p)], \tag{5.2}$$

$$\cdots\cdots\quad\cdots\cdots\quad\cdots\cdots,$$

$$\psi_{2i}(p)=p^{(k-2)}\phi_i[p\,\phi_i(p)].$$

This sequence of i renormalization steps corresponds to the renormalization of one full fractal segmentation step. Then such a sequence is iterated as many times as the total number k of segmentation steps of the fractal. Let us define the function $g_{n,j}(p)$ by

$$g_{n,0}(p)=p,\ \text{ and }\ g_{n,j+1}(p)=p\,\phi_i[g_{n,j}(p)] \tag{5.3}$$

the jth iteration $(1\leqslant j<k)$ of this sequence of i renormalization steps gives i more successive renormalized concentrations as follows:

TABLE I. The fixed points $p_{n,k}$ of RSC(n,q,k), for $4\leqslant n\leqslant 64$, and for $k\leqslant 2\leqslant 11$ and $k=\infty$.

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=\infty$
$p_{4,k}$	0.729 21	0.772 09	0.794 39	0.807 81	0.816 62	0.822 76	0.827 23	0.8306	0.8332	0.835 25	0.848
$p_{8,k}$	0.705 27	0.737 69	0.754 06	0.763 65	0.7698	0.773 99	0.776 99	0.779 21	0.7809	0.782 22	0.7899
$n_{16,k}$	0.685 06	0.708 78	0.720 37	0.726 99	0.731 16	0.733 95	0.735 92	0.737 36	0.738 44	0.739 28	0.744
$p_{32,k}$	0.668 59	0.6855	0.6935	0.697 97	0.700 74	0.702 57	0.703 84	0.704 77	0.705 46	0.706	0.708 92
$p_{64,k}$	0.655 55	0.667 37	0.6728	0.675 77	0.677 58	0.678 77	0.6796	0.680 19			0.682 79

TABLE II. The thermal exponent $\nu_{n,k}$ of RSC(n,q,k), for $n=8$, and $n=16$, and $k\leqslant 2\leqslant 5$ and $k=\infty$.

	$k=2$	$k=3$	$k=4$	$k=5$	$k=\infty$
$\nu_{8,k}$	1.985 26	2.338 65	2.6328	2.869 04	4.019 49
$\nu_{16,k}$	1.851 01	2.013 89	2.122 65	2.196 45	2.439 01

$$\begin{aligned}
&\psi_{[(j-1)i+1]}(p)=p^{(k-j+1)}\phi[g_{n,j-1}(p)],\\
&\psi_{[(j-1)i+2]}(p)=p^{(k-j+1)}\phi_2[g_{n,j-1}(p)],\\
&\cdots\cdots\quad\cdots\cdots\quad\cdots\cdots,\\
&\psi_{ji}(p)=p^{(k-j)}g_{n,j}(p)
\end{aligned}\tag{5.4}$$

and the final renormalized concentration, obtained after $k\times i$ renormalization steps is

$$\psi_{ki}(p)=\phi_i[g_{n,k-1}(p)]. \tag{5.5}$$

Then, the full renormalization of RSC(n,q,k) is divided into k sequences of i renormalization steps. One sequence corresponds to the full renormalization of one fractal segmentation step, which is similar to that of q independant regular square lattices. This result explains the three features observed in the numerical simulations: (i) we clearly obtain k successive fixed points, corresponding to the renormalization of each segmentation step of the fractal; (ii) the jth fixed point $(1\leqslant j\leqslant k)$ is the intersection of the $i+1$ following functions: $\psi_{(j-1)i}$, $\psi_{(j-1)i+1}$, $\psi_{(j-1)i+2}$, ... ψ_{ji}. Then, each fixed point involves exactly $i+1$ renormalization steps; (iii) the values of the fixed points are not changed when k increases, only new points are added.

For comparison to the numerical simulations, Figs. 6 and 7 show the curves obtained for the same cases as Figs. 4 and 5, respectively, i.e., RSC$(8,q,4)$ and RSC$(16,q,3)$. Theoretical and numerical results are in good agreement. Only a slight discrepancy shows up, due to finite size effects. The function ϕ is indeed calculated in the limit $n\to\infty$, and used in the previous examples for finite values of n: 8 and 16.

The sequence of fixed points $p_{n,k}$ is increasing with k. The kth fixed point $p_{n,k}$ can be considered as the percolation threshold of RSC(n,q,k), and is calculated as follows. It is the real solution between 0 and 1 of the equation

$$\psi_{(k-1)i}(p)=\psi_{(k-1)i+1}(p)\quad\text{or}\quad g_{n,(k-1)}(p)=p_c. \tag{5.6}$$

When $k\to\infty$, $g_{n,k}$ converges towards a limit g_n and $p_{n,k}$ towards a limit p_n strictly less than 1, which can be considered as the percolation threshold of RSC(n,q,∞) (independent of any segmentation step k). These percolation thresholds p_n are calculated as follows. From the recurrence 5.3 on $g_{n,k}$, g_n obeys the equation

$$g_n(p)=p\,\phi_i[g_n(p)] \tag{5.7}$$

for $n=2$ $(i=1)$, the threshold is exactly calculated

$$p_2=\frac{3}{4}\left(\frac{3}{2}\right)^{1/2}. \tag{5.8}$$

However, for such a small value of n, the finite size effects are too strong to allow any comparison with numerical simulations. For higher values of n, the degrees of the equations are higher than four and the thresholds are calculated numerically. Table I contains the values of $p_{n,k}$ for $1\leqslant i\leqslant 6$ $(4\leqslant n\leqslant 64)$ and $2\leqslant k\leqslant 11$, and $k=\infty$. For $k=1$, all fixed points are equal to $p_c=(\sqrt{5}-1)/2$, since finite size effects are neglected.

Figure 8 shows a plot of $p_{n,k}$, and p_n versus n for $2\leqslant i\leqslant 6$ $(2\leqslant n\leqslant 64)$ and $2\leqslant k\leqslant 11$ and $k=\infty$. It results from the calculation of $p_{n,k}$ and p_n that both converges towards p_c, the percolation threshold of the lacunary square lattice, when $n\to\infty$ as expected from physical reasons.

An estimate of the thermal exponent $\nu_{n,k}$ which characterizes the behavior of the correlation length ξ at large scale: $\xi\sim(p-p_c)^{-\nu}$ is obtained by linearization of the renormalization function near the fixed points $\nu=\ln(2)/\ln(d\phi/dp_{/p=p_c})$. In the fractal case, the result is a sequence of thermal exponents depending on the segmentation step order k (i.e., on the scale)

$$\nu_{n,k}=\frac{\ln(2)}{\ln\left(\dfrac{d\phi}{dp}{}_{/p=p_{n,k}}\right)}. \tag{5.9}$$

The results for $n=8$ and $n=16$ are shown in Table II. The sequence $\nu_{n,k}$ increases with k, decreases with n, and converges, when $n\to\infty$, towards 1.635 28, the value of the exponent ν obtained by renormalization of the infinite regular lacunary square lattice.

VI. CONCLUSION AND PROSPECTS

The main result of this work is that, in self-similar structures, the percolation threshold may depend on the scale of observation. We suggest an experimental check, in ''natural'' self-similar fractals of this scale dependence of the percolation threshold. A classic test of percolation is to check whether or not a liquid can go through a porous media. A simple liquid has no particular scale (except at the molecular size, which is not our aim), but it is possible to give it a scale by putting in suspension particules of a calibrated size. We expect that, in a self-invariant structure, and when the size of the particules decreases, more and more holes are available, so that it is more and more easy to percolate. Then the concentration of the fractal at the percolation threshold, and at the corresponding scale (i.e., the corresponding size) should also decrease.

This is in good agreement with our theoretical results. This clearly appears on the plot of the concentrations of RSC at the percolation thresholds versus the scale invariance ratio n and the scale k in Fig. 9 (in our case, the occupied sites of our model corresponds to the holes of the porous media).

The physical applications of such flows of solid particles immersed in a fluid are numerous in industry, in the fields of filtration, chromatography, clogging, and blood circulation.

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